## Exam Graph Theory, 29 January 2005

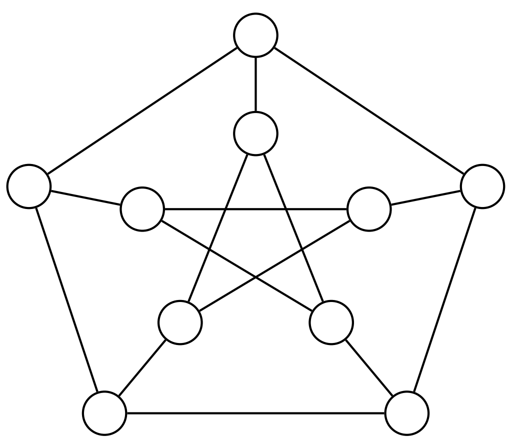
**1** [**15**]

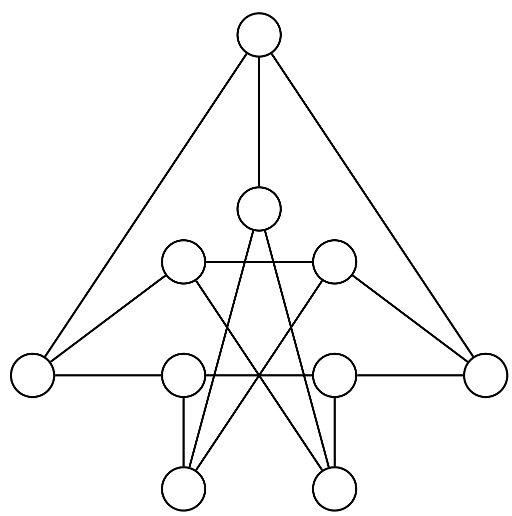
a. Draw a graph with an even number of vertices with one vertex of degree 1 and all other vertices of degree 3, or explain why such a graph does not exist.

b. Determine the minimum number of vertices needed for a 3-regular graph, and draw such a graph.

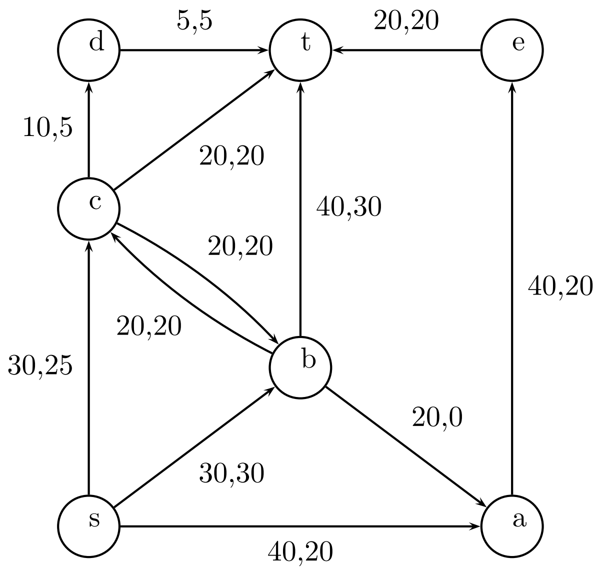
c. Does there exist a 5-regular graph with 10 vertices? Explain!

**2** [**15**] Are the two graphs below isomorphic? Either indicate an isomorphism or explain why the graphs are not isomorphic.





**3** [**15**] In the network below, the first number near an arc represents the capacity of the arc, and the second number represents the flow through the arc. Vertex $s$ represents the source and $t$ represents the sink.

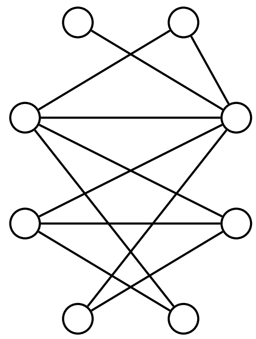



a. Determine a maximum flow, starting with the given flow as the initial flow. Clearly show your work for every iteration of the algorithm.

b. Determine a minimum cut for the network.

**4** [**10**]

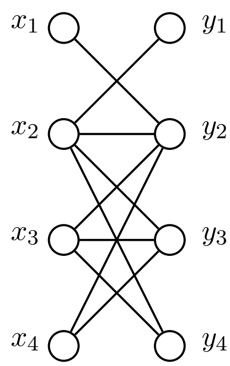
a. Determine a DFS-labelling for the graph below. Start the labelling process at the vertex of degree 1. Also assign numbers (consecutively, $1, 2, \ldots$) to the edges of the graph in the order the edges were visited during the DFS-labelling. Edges that were not visited during the DFS-labelling should also be assigned consecutive numbers. Choose those edges in arbitrary order.



b. Assign a weight to every edge equal to 12 minus its number. Determine a minumum cost spanning tree with Kruskal's algorithm. Show clearly what happens at every iteration.

**5** [**10**]

a. Determine a maximal matching that is not maximum for the graph below.




b. Consider the matching $M : x_2 - y_1,\ x_3 - y_4,\ x_4 - y_2$. Determine an $M$-alternating path and, using this path, determine a matching with one extra pair.

c. Does there exist a complete matching that has no edge in common with $M$. If so, give one. If not, explain why not.

**6** [**15**] Prove that a graph is Hamiltonian if it contains only one cycle and if the degree of every vertex is at least 2.

**7** [**10**] Consider a set of $n$ countries that are numbered from 1 to $n$. The numbering is such that if two countries $i$ and $j$ share a common border, then the set of countries with an index between $i$ and $j$ ($i$ and $j$ not included) can only share borders with each other, or with $i$ or $j$.

a. Give three examples, for $n = 3$, $n = 4$ and $n = 5$, to show that the number of borders between pairs of countries can be as high as $2n - 3$.

b. The number $2n-3$ is indeed an upper bound for the number of borders between pairs of countries. Prove this.