Exam Linear Algebra, December 22, 2004

1 Consider the following system of linear equations.

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & + & x_3 & & & = & -2 \\ 2x_1 & & & + & x_3 & + & 3x_4 & = & 4 \\ & & 2x_2 & - & x_3 & + & 3x_4 & = & 8 \\ 3x_1 & - & x_2 & & & + & x_4 & = & 6 \end{array}$$

a. [**10 points**] Give all solutions of this system.

Solution Put the system in matrix-form and perform row-reduction.

$$\begin{pmatrix} 1 & -1 & 1 & 0 & -2 \\ 2 & 0 & 1 & 3 & 4 \\ 0 & 2 & -1 & 3 & 8 \\ 3 & -1 & 0 & 1 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 & 0 & -2 \\ 0 & 2 & -1 & 3 & 8 \\ 0 & 2 & -1 & 3 & 8 \\ 0 & 2 & -3 & 1 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 & 0 & -2 \\ 0 & 2 & -1 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 & 0 & -2 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} & 4 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

All solutions:

$$\begin{array}{rcrcrl} x_1 & = & 3 & - & x_4 & \\ x_2 & = & 3 & - & 2x_4 & \\ x_3 & = & -2 & - & x_4 & . \\ x_4 & = & & & x_4 & \text{(free variable)} \end{array}$$

b. [**10 points**] Determine a basis for the set

$$\{\begin{pmatrix} x_1 - x_2 + x_3 \\ 2x_1 + x_3 + 3x_4 \\ 2x_2 - x_3 + 3x_4 \\ 3x_1 - x_2 + x_4 \end{pmatrix} \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\}.$$

Solution The set consists of all vectors of the form

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 3 \\ 3 \\ 1 \end{pmatrix}.$$

So it is the row space of the matrix

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 2 & 0 & 1 & 3 \\ 0 & 2 & -1 & 3 \\ 3 & -1 & 0 & 1 \end{pmatrix}.$$

The necessary calculations were done in 1a: the first three columns of this matrix have a pivot. Therefore, a basis for the set is

$$\{\begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}\}.$$

2. True or false? If the statement is true, explain why. If the statement is false, give a counterexample.

a. [**4 points**] The matrix $\begin{pmatrix} k & -3 \\ 4 & k-8 \end{pmatrix}$ is invertible for all $k \in \mathbb{R}$.

Solution The determinant equals

$$k(k-8) + 12 = k^2 - 8k + 12 = (k-2)(k-6).$$

The determinant is zero for $k = 2$ and $k = 6$. For these two values of k the matrix is not invertible. Therefore, the statement is false.

b. [**4 points**] If the matrices A and B are invertible, then AB is invertible (assuming that AB exists).

Solution One can easily check that $ABB^{-1}A^{-1} = I$, so $B^{-1}A^{-1}$ is the inverse of AB. Therefore, the statement is true.

c. [**4 points**] If the matrix A is diagonalizable, then A^2 is diagonalizable.

Solution If A is diagonalizable, then $A = PDP^{-1}$, where D is a diagonal matrix. Then $A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$, and we see that A^2 can be written in diagonal form. Therefore, the statement is true.

d. [**4 points**] If v_1, v_2, v_3 are non-zero vectors such that $v_1^T v_2 = v_1^T v_3 = v_2^T v_3 = 0$, then $\{v_1, v_2, v_3\}$ is a linearly independent set.

Solution The statement is true, since each pair of vectors is orthogonal. (Theorem 4 from paragraph 6.2.)

e. [**4 points**] If $\{v_1, v_2\}$, $\{v_1, v_3\}$ and $\{v_2, v_3\}$ are all linearly independent sets, then $\{v_1, v_2, v_3\}$ is also a linearly independent set.

Solution The statement is false. A counterexample is given by $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $v_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

f. [**4 points**] A system of linear equations with more variables than equations has an infinite number of solutions.

Solution The statement is false. A counterexample is given by

$$\begin{array}{ccccccccc} u & + & v & + & w & + & x & = & 0 \\ & & & & & & x & = & 1 \\ & & & & & & x & = & 2 \end{array}.$$

(The system is not consistent.)

g. [**4 points**] For every matrix A, the matrix AA^T is symmetric.

Solution A matrix X is symmetric if $X = X^T$. Now, the statement is true, since

$$(AA^T)^T = (A^T)^T A^T = AA^T.$$

3. The matrix A is given by

$$A = \begin{pmatrix} 2 & 1 & -3 & 4 & -1 \\ 2 & 2 & -4 & 5 & -1 \\ 4 & -3 & -1 & 3 & -2 \end{pmatrix}$$

a. [**10 points**] Determine a basis for the rowspace Row(A) and the columnspace Col(A). Also, determine the rank $r(A)$.

Solution These question can be answered by row reduction:

$$\begin{pmatrix} 2 & 1 & -3 & 4 & -1 \\ 2 & 2 & -4 & 5 & -1 \\ 4 & -3 & -1 & 3 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & -3 & 4 & -1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & -5 & 5 & -5 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & -3 & 4 & -1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -0 & 0 \end{pmatrix}$$

The pivots are in the first two columns of the last matrix. Then the first two columns of A form a basis for the column space of A.

There are two non-zero rows in the last matrix. These two rows form a basis for the row space of A.

There are two pivots, so the rank of A is 2.

b. [**10 points**] Let v_1, v_2 and v_3 denote the rows of the matrix A. Determine numbers α_1, α_2 and α_3, not all of them equal to 0, such that $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$.

Solution We wish to find numbers α_1, α_2 and α_3 such that

$$\alpha_1 \begin{pmatrix} 2 \\ 1 \\ -3 \\ 4 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 2 \\ -4 \\ 5 \\ -1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 4 \\ -3 \\ -1 \\ 3 \\ -2 \end{pmatrix} = 0.$$

We can also write this vector equation in matrixform:

$$\begin{pmatrix} 2 & 2 & 4 \\ 1 & 2 & -3 \\ -3 & -4 & -1 \\ 4 & 5 & 3 \\ -1 & -1 & -2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0.$$

We solve it by row reduction:

$$\begin{pmatrix} 2 & 2 & 4 \\ 1 & 2 & -3 \\ -3 & -4 & -1 \\ 4 & 5 & 3 \\ -1 & -1 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & -3 \\ -3 & -4 & -1 \\ 4 & 5 & 3 \\ -1 & -1 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & -1 & 5 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

All solutions are

$$\begin{array}{rcr} \alpha_1 & = & -7\alpha_3 \\ \alpha_2 & = & 5\alpha_3 \\ \alpha_3 & = & \alpha_3 \end{array} .$$

A non-zero solution is found by taking $\alpha_3 = 1$. We find

$$\begin{array}{rcr} \alpha_1 & = & -7 \\ \alpha_2 & = & 5 \\ \alpha_3 & = & 1 \end{array} .$$

4. Consider the matrix A given by

$$A = \begin{pmatrix} 13 & 4 \\ 4 & 7 \end{pmatrix}$$

a. [**9 points**] Determine the eigenvalues of A.

Solution The characteristic polynomial is

$$(13-\lambda)(7-\lambda)-16 = 91-7\lambda-13\lambda+\lambda^2-16 = \lambda^2-20\lambda+75 = (\lambda-15)(\lambda-5).$$

For $\lambda = 15$ and $\lambda = 5$, the polynomial is zero. Therefore, the eigenvalues are 15 and 5.

b. [**9 points**] For each eigenvalue, determine a basis for its corresponding eigenspace.

Solution For $\lambda = 15$, the eigenspace is given by the solutions to the system

$$\begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

Row reduction:

$$\begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix} \to \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}.$$

The solutions are given by $x_1 = 2x_2$, x_2 free. The vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, for example, is a basis for this set.

For $\lambda = 5$, the eigenspace is given by the solutions to the system

$$\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

Row reduction:

$$\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}.$$

The solutions are given by $x_1 = -\frac{1}{2}x_2$, x_2 free. The vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, for example, is a basis for this set.

c. [**5 points**] Determine matrices Q and D such that $A = QDQ^T$.

Solution This question was not well-posed, since it has the correct, but trivial answer $Q = I$ and $D = A$. If we require D to be a diagonal matrix, then we take

$$D = \begin{pmatrix} 15 & 0 \\ 0 & 5 \end{pmatrix},$$

and for Q we take the matrix whose columns are eigenvectors of length 1 (one from each eigenspace), i.e.

$$Q = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{pmatrix}$$

d. [**9 points**] Without doing any calculations, give 4 linearly independent eigenvectors of the matrix M given by

$$M = \begin{pmatrix} 13 & 4 & 0 & 0 \\ 4 & 7 & 0 & 0 \\ 0 & 0 & 13 & 4 \\ 0 & 0 & 4 & 7 \end{pmatrix}.$$

Solution The eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for A gives rise to two eigenvectors for M:

$$\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

(Also any linear combination, like $\begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector for M, but such vectors are dependent on the first two.)

The eigenvector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ for A gives rise to two more linearly independent eigenvectors for M:

$$\begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \end{pmatrix}.$$